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Submartingale property of subharmonic functions.

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In this note we consider conditions for the validity of submartingale property and L^1-Liouville theorems for subharmonic function on Riemannian manifolds. Let M be a complete and stochastically complete Riemannian manifold. X_t denotes Brownian motion on M with non-random initial point $X_0 \in M$. We say that submartingale property holds for u if $u(X_t)$ is a submartingale for all initial point $X_0 \in M$. It is known that there exist manifolds which allow subharmonic functions without satisfying submartingale property(see section 4).

We ask when manifolds admit subharmonic functions in a suitable class having submartingale property and what a geometrical meaning of submartingale property is.

We also note a relationship between the submartingale property for integrable subharmonic functions and L^1-Liouville property.

1 A simple and general observation on submartingale property

Define $\mathcal{U}$ be a collection of the non-negative and locally Lipschitz continuous functions such that if $u \in \mathcal{U}$, then Δu is a nonnegative smooth measure and $E_x[\int_0^t \Delta u(X_s)ds] < \infty$ for all $0 \leq t < \infty$ and $x \in M$.

Define a *default function* $N_x(T, u)$ for a function u and a stopping time T by

$$N_x(T,u) = \lim_{\lambda \to \infty} \lambda P_x(\sup_{0 \leq s \leq T} u(X_s) > \lambda).$$

By Ito's formula or Fukushima decomposition it is easy to see

Proposition 1 (Elworthy-X.M.Li-Yor [4], [5]). *Suppose $u \in \mathcal{U}$. If $N_x(t,u) = 0$ $(\forall t > 0)$, then $u(X_t)$ is a submartingale under P_x and*

$$E_x[u(X_t)] - u(x) = \frac{1}{2} E_x[\int_0^t \Delta u(X_s)ds].$$

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Remark. Assume $u \in \mathcal{U}$. If $N_{x_o}(t,u) = 0$ for some $x_o \in M$, then $N_x(t,u) = 0$ for any $x \in M$ since X_t has a continuous heat kernel on M. If $N_x(t_o,u) = 0$ for some $t_o > 0$ and all $x \in M$, then $N_x(t,u) = 0$ for any $t > 0$ by Markovian property.

The default function was considered first by Carne([3]) in his probabilistic interpretation of classical Nevanlinna theory and the author([1]) discussed a generalization for a higher dimensional setting.

Takaoka([15]) considered a condition for a continuous local martingale to be a pure martingale using this default function.

Elworthy-X.M.Li-Yor([4],[5]) emphasized "the importance of strictly local martingales" using this default function and gave some applications to radial Ornstein-Uhlenbeck processes.

Let $B_x(r)$ denote the geodesic ball of radius r with center x and $\tau_r = \inf\{t > 0 : X_t \notin B_x(r)\}$. It is easy to see

Lemma 2.

$$\lim_{r \to \infty} E_x[u(X_{\tau_r}) : \tau_r < t] = N_x(t,u)$$

for $u \in \mathcal{U}$.

Hence we can use the following estimate due to M. Takeda.

Lemma 3 (Takeda's inequality([16], see also [6])).

$$\int_{B_x(1)} P_y(\tau_r < t)dv(y) \leq const. \frac{vol(B_x(r+1))}{r} e^{-\frac{cr^2}{t}},$$

where dv *denotes the Riemannian volume measure on* M.

Remark. This estimates holds for general symmetric diffusions.

Let $M_x(r) = \sup_{y \in \partial B_x(r)} u(y)$.

Directly from Takeda's inequality

Proposition 4. *If* $u \in \mathcal{U}$ *and*

$$\liminf_{r \to \infty} \frac{1}{r^2}(\log M_x(r) + \log vol(B_x(r))) < \infty$$

for some $x \in M$, *then* $u(X_t)$ *is a submartingale under* P_x *for all* $x \in M$.

2 A condition for subharmonic functions and manifolds to have submartingale property with Ricci curvature

Let

$$R(x) = \inf_{\xi \in T_x M,\ ||\xi||=1} Ric(\xi, \xi).$$

To estimate $M_x(r)$, we can use an estimate by P.Li and R.Schoen:

Lemma 5. *[P.Li-Schoen[9]] Assume M satisfies that $R(x) \geq -\kappa(r(x))$ for a nondecreasing function $\kappa(r) \geq 0$. Let u be a nonnegative smooth subharmonic function. Then there exists constants $C_1 > 0$, $C_2 > 0$ such that*

$$\max_{x \in \partial B(r/2)} u(x) \leq C_1 e^{C_2 r \sqrt{\kappa(5r)}} \mathrm{Vol}(B(r))^{-1} \int_{B(r)} u dv.$$

Remark. A similar estimate to the above for subharmonic functions of uniformly elliptic diffusion operator on $\mathbf{R}^n$ is given in Saloff-Coste's monograph([13]).

Directly this with Proposition 4 implies

Proposition 6. *Assume $R(x) \geq -k(r(x))$ for nonnegative continuous increasing function k on $[0, \infty)$ and $r(x) = d(o, x)$ for some $o \in M$. If $u \in \mathcal{U}$ is smooth and*

$$\liminf_{r \to \infty} \frac{1}{r^2} (\log k(r) + \log \int_{B_o(r)} u(x) dv(x)) < \infty$$

for some $o \in M$, then $u(X_t)$ is a submartingale[†].

Remark. The condition

$$\liminf_{r \to \infty} \frac{1}{r^2} \log k(r) < \infty$$

ensures stochastic completeness of M.

For some applications we wish to replace

$$\int_{B_o(r)} u(x) dv(x) \text{ by } \int_{B_o(r)} \Delta_M u(x) dv(x).$$

Theorem 7. i) *Assume $R(x) \geq -cr(x)^2 - c$ for $c \geq 0$.*
If $u \in \mathcal{U}$ is smooth and

$$\liminf_{r \to \infty} \frac{1}{r^2} \log^+ \int_{B_o(r)} \Delta_M u(x) dv(x) < \infty$$

for some $o \in M$, then $u(X_t)$ is a submartingale.
ii) *Assume $R(x) \geq -k(r(x))$ for a nonnegative nondecreasing function k such that $\lim_{r \to \infty} k(r)/r^2 = 0$. If*

$$\int_0^\infty e^{-\epsilon r^2} \int_{B_o(r)} \Delta_M u(x) dv(x) dr < \infty \ (\forall \epsilon > 0)$$

for some $o \in M$, then $u(X_t)$ is a submartingale.

[†]When there is no danger of confusion, we omit 'under P_x' from now on.

Corollary 8. *Assume $R(x) \geq -cr(x)^2 - c$ for $c \geq 0$.*
If u is a positive harmonic function, then $u(X_t)$ is a martingale.

The proof of Theorem 7 is derived by the Green's formula and coarea formula. In fact we have the following.

Lemma 9. *([2]) Suppose that M satisfies $R(x) \geq -cr(x)^2 - c$ for $c \geq 0$ and a nonnegative smooth subharmonic function u satisfies*

$$\int_{B_o(r)} \Delta u(x)dv(x) = O(e^{c_0 r^2}) \text{ for a constant } c_0 > 0.$$

Then we have
i)

$$\int_{B_o(r)} u(x)dv(x) = O(e^{c_1 r^2}) \quad \text{a.e. } r \in (0,\infty)$$

and
ii)

$$\max_{x\in\partial B_o(r)} u(x) = O(e^{c_2 r^2}) \quad \text{a.e. } r \in (0,\infty)$$

for some positive constants c_1, c_2.

3 L^1-Liouville theorems

We say M has L^p-Liouville property if any non-negative smooth subharmonic function L^p-integrable with respect to the Riemannian volume measure is constant.

It is easy to see that

Proposition 10. *The following two properties of M are equivalent.*
i) *Every nonnegative and integrable subharmonic function on M has submartingale property.*
ii) *M has L^1-Liouville property.*

Proof. ii) $\Rightarrow$ i) : trivial.

i) $\Rightarrow$ ii) : Let u be a nonnegative, smooth subharmonic function on M. Submartingale property of u means

$$u(x) \leq E_x[u(X_t)]$$

for all $0 < t$ and $x \in M$. Then

$$tu(x) \leq \int_0^t E_x[u(X_s)]ds.$$

If X is recurrent, ratio ergodic theorem for recurrent Markov processes(cf.[12]) implies

$$\frac{1}{t}E_x[\int_0^t u(X_s)ds] \to \begin{cases} \frac{\int_M u(x)dx}{vol(M)} & (\text{ if } vol(M) < \infty), \\ 0 & (\text{ if } vol(M) = \infty) \end{cases}$$

as $t \to \infty$. In both cases u should be bounded. Then u is a constant.

If X is transient, $\frac{1}{t}E_x[\int_0^t u(X_s)ds] \to 0$ as $t \to \infty$ since $E_x[\int_0^\infty u(X_s)ds] < \infty$ for an integrable function u. □

Then we recover P.Li's L^1-Liouville theorem.

Theorem 11. *[P.Li([8])] Assume M is a geodesically complete Riemannian manifold and $R(x) \geq -cr(x)^2 - c$ for $c \geq 0$. Then M has L^1-Liouville property.*

Remark 1. This theorem is improved by X.D.Li in [10] to the case when $L = \Delta - \nabla\phi \cdot \nabla$ with modified Ricci curvature.

Remark 2. It is well-known that L^p-Liouville property for $p > 1$ holds for any complete Riemannian manifolds. This is due to S.T.Yau([17]). This is improved by K.T.Sturm([14]) under the setting of symmetric diffusions.

Remark 3. For $p = 1$ there are few results except for ones due to P.Li and Nadirashvili. Nadirashvili obtained the following result:

Theorem 12 (Nadirashvili([11])). *Let M be a geodesically complete Riemannian manifold and u a smooth non-negative subharmonic function.*

i) *If $\int_M \frac{f(u)}{1+r(x)^2}dv(x) < \infty$ with a non-negative increasing function f on $[0, \infty)$ satisfying $\int^\infty \frac{1}{f(t)}dt < \infty$, then u is a constant.*

ii) *If $\int_M u(x)dv(x) < \infty$ and $u(x) = O(e^{r(x)^{2-\epsilon}})$ for some $\epsilon > 0$, then u is a constant.*

Remark. Examples of f in i): $f(x) = x^p$ $(p > 1)$, $f(x) = x(\log x)^p$ $(p > 1)$ etc.

4 Examples

1. Remark that if $Ric_M \geq 0$ or M is simply connected and of non-positive constant curvature, then $u(X_t)$ is a submartingale for $u \in \mathcal{U}$.

2. The following example is originally due to Li-Schoen. Let $\overline{M}$ be a compact 2-dim Riemannian manifold with a metric ds_0^2 and $\overline{X}$ Brownian motion on $\overline{M}$. Fix $o \in \overline{M}$. Set

$$g(o, x) = 2\pi \int_0^\infty (p(t, o, x) - \frac{1}{vol(\overline{M})})dt + C,$$

where $p(t, x, y)$ is the transition density of $\overline{X}$ and C is a positive constant such that $g(o, x) > 0$ for all $x \in \overline{M}$. Remark that $g(x, y) \sim \log \frac{1}{d(x,y)^2}$ $(d(x, y) \to 0)$. Note $\frac{1}{2}\Delta_{\overline{M}} g(o, x) = -2\pi\delta_o(x) + \frac{1}{Vol(\overline{M})}$.

Let M be $\overline{M} \setminus \{o\}$. Take σ be a smooth function on M s.t.

$$\sigma(x) \sim t^{-1}(\log \frac{1}{t})^{-1}(\log\log \frac{1}{t})^{-\alpha} \text{ with } 1/2 < \alpha < 1$$

when $t = d_{\overline{M}}(o, x) \to 0$.

Define a metric $ds^2 = \sigma^2 ds_0^2$ on M. Note that Laplacian Δ_M defined from ds^2 has a form

$$\Delta_M = \sigma^{-2}\Delta_{\overline{M}},$$

where $\Delta_{\overline{M}}$ is defined from ds_0^2.

(M, ds^2) satisfies

- complete.
- M is of finite volume w.r.t ds^2.
- u is a nonnegative smooth subharmonic function on M and integrable w.r.t. ds^2.
- the curvature $\sim -const.r^{\frac{2\alpha}{1-\alpha}} = -cr^{2+\epsilon}$ as $r \to \infty$ ($\epsilon = (4\alpha - 2)/(1 - \alpha) > 0$).

Remark. $u(\overline{X}_t)$ is not a submartingale and (M, ds_o^2) is incomplete but stochastically complete.

This example shows that it is difficult to improve the condition on Ricci curvature in Theorem 11. Also L^1-Liouville property can not be controled only by the volume growth of manifolds.

5 Another criterion and weighted L^1-Liouville theorem

In this section we consider another setting and discussed the validity of weighted L^1-Liouville theorems.

We will assume later that

(*) M has a nonnegative subharmonic exhaustion function ϕ such that $|\nabla\phi|$ is bounded on M.

Example: Let $\iota : M \to \mathbf{R}^n$ minimally and properly immersed manifold and $\phi(x) := d(o, \iota(x))$ with Euclidean distance $d(o, y)$. Then ϕ satisfies above condition and $\phi \in \mathcal{U}$ ($\Delta\phi$ is bounded) w.r.t the induced metric.

Remark: 1. Any Stein manifold can be properly emmbedded in $\mathbf{C}^m$ and any complex submanifold in complex Euclidean space satisfies the above.

2. We do not assume here that $\phi \in \mathcal{U}$. If $\Delta\phi$ is bounded, then M is stochastically complete.

3. Every complete Riemannian manifold has a nonnegative, smooth, subharmonic exhaustion function (Greene-Wu [7]).

Let us consider estimates on the Poisson kernel $P_r(x,y)$ on $B_o(r)$ and Green's function $g_r(x,y)$ on $B_o(r)$ of Δ_M with Dirichlet boundary condition on $\partial B_o(r)$, where $B_x(r) = \{y \in M | \phi(y) - \phi(x) < r\}$.

Lemma 13. *Assume that ϕ is a nonnegative, smooth subharmonic exhaustion function on M.*

If $\phi(x) < \alpha < r$, for $y \in \partial B_o(r)$

$$P_r(x,y) \leq \frac{\sup_{w\in\partial B_o(\alpha)} g_r(x,w)}{r-\alpha} |\nabla\phi|(y).$$

Note that if $\phi(x) < \alpha < \beta < r$,

$$\frac{\sup_{w\in\partial B(\beta)} g_r(x,w)}{r-\beta} \leq \frac{\sup_{w\in\partial B(\alpha)} g_r(x,w)}{r-\alpha} < \infty$$

and $\lim_{\alpha\to\phi(x)} \frac{\sup_{w\in\partial B(\alpha)} g_r(x,w)}{r-\alpha} = \infty$.

We define a new quantity $c(r)$ by

$$c(r) = \sup_{x\in\partial B(r/2)} (\lim_{\beta\to r} \frac{\sup_{w\in\partial B(\beta)} g_r(x,w)}{r-\beta}) < \infty.$$

Then

Lemma 14.

$$\sup_{x\in\partial B_o(r/2)} u(x) \leq c(r) \sup_{z\in B_o(r)} |\nabla\phi|^2(z) \int_{\partial B_o(r)} u(y) \frac{dA_r(y)}{|\nabla\phi|(y)},$$

where dA_r is the induced volume form on $\partial B_o(r)$.

The above $c(r)$ plays the same role as the bounds of Ricci curvature in P.Li-Schoen estimate. We have the following results with using $c(r)$.

Theorem 15. *Assume (*). If $u \in \mathcal{U}$ and*

$$\liminf_{r\to\infty} \frac{1}{r^2} \left(\log c(r) + \log vol(\{\phi(x) < r\}) + \log \int_{\{\phi(x)<r\}} u(x) dv(x) \right) < \infty,$$

then $u(X_t)$ is a submartingale.

We also have a weighted L^1-Liouville theorem as follows.

Theorem 16. *Assume (*).*
i) *Assume* $\liminf_{r\to\infty} \frac{1}{r^{2(1-p)}} (\log c(r) + \log vol(\{\phi(x) < r\})) < \infty$ *for* $0 \leq p < 1$. *If*

$$\int_M \frac{u(x)}{(1+\phi(x))^{2p}} dv(x) < \infty,$$

then $u = 0$.
ii) *Assume* $\liminf_{r\to\infty} \frac{1}{(\log r)^2} (\log c(r) + \log vol(\{\phi(x) < r\})) < \infty$.
If

$$\int_M \frac{u(x)}{1+\phi(x)^2} dv(x) < \infty,$$

then $u = 0$.

Proof. Use Takeda's inequality and time change argument with an estimate in Lemma 14.

We apply our discussion to some simple cases.

Theorem 17. *Assume M is a complete Riemannian manifold of finite volume such that ϕ is an exhaustion function with bounded $|\nabla\phi|$. If a nonnegative and smooth subharmonic function u satisfies*

$$u(x) = O(\phi(x)^2) \qquad (x \to \infty),$$

then u is constant.

Remark. In the proof of the above result the growth of u enables us to skip the estimate in Lemma 14.

We can easily to check that these results hold in the case when X is a symmetric diffusion on a smooth manifold M. We employ usual setting of symmetric diffusions as follows(see [6] for details). X has a generator L on $L^2(M; dm)$ where dm is a Radon measure on M. The square field operator $\Gamma(\phi, \phi)$ can be defined by

$$2\Gamma(\phi, \phi) = L\phi^2 - 2\phi L\phi \quad \text{for } \phi \in C_0^\infty(M).$$

This is a bilinear operator. The corresponding Dirichlet form to X takes a form as

$$\mathcal{E}(\phi, \phi) = \frac{1}{2} \int_M d\Gamma(\phi, \phi).$$

Remark that $d\Gamma(\phi, \phi)$ is a Radon measure on M for general ϕ belongs locally to the domain of $\mathcal{E}$. When X is Brownian motion on a Riemannian manifold M, $L = \frac{1}{2}\Delta_M$, $dm = dv$ and $\Gamma(\phi, \phi) = |\nabla\phi|^2$ for $\phi \in C_0^\infty(M)$. Then we say that u is L-subharmonic if $Lu = 0$ in distribution sense. Replacing Riemannian quantities like dv and norm of gradient by the quantities in this diffusion setting like dm and squre field operator, we have a simple generalization of the above results. We say that $\Gamma(\phi, \phi)$ is bounded if $d\Gamma(\phi, \phi) \leq const.dm$. We have the following.

Theorem 18. *Assume that M has a nonnegative exhaustion function ϕ whose $\Gamma(\phi, \phi)$ is bounded and M satisfies $m(M) < \infty$. If a nonnegative and smooth L-subharmonic function u satisfies*

$$u(x) = O(\phi(x)^2) \qquad (x \to \infty),$$

then u is constant.

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